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Adjoints of Semigroups of Linear Operators in Banach Spaces

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1. INTRODUCTION

Let X be a Banach space with dual X^* and let $B(X)$ be the set of all bounded linear operators on X. We denote by $\langle x, x^* \rangle$ the value of $x^* \in X^*$ at $x \in X$. A one-parameter family $\{T(t) : t \geqslant 0\}$ in $B(X)$ is called a *semigroup on* X if

$$T(t+s) = T(t)\,T(s) \quad \text{for} \quad t, s \geqslant 0 \quad \text{and} \quad T(0) = I \text{ (the identity)}, \tag{1.1}$$

$$\text{for each } x \in X,\ T(t)x \text{ is continuous in } t > 0. \tag{1.2}$$

If a semigroup $\{T(t) : t \geqslant 0\}$ satisfies the condition that $\lim_{t \to 0^+} T(t)x = x$ for $x \in X$ then it is called a *semigroup of class* (C_0) on X (see [2, 9]).

Let $\{T(t) : t \geqslant 0\}$ be a semigroup on X. We define the infinitesimal generator A_0 by

$$A_0 x = \lim_{h \to 0^+} h^{-1}(T(h)x - x) \tag{1.3}$$

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whenever the limit exists. If A_0 is closable then the closure $\overline{A_0}$ of A_0 is called the *complete infinitesimal generator* of the semigroup. The notion of the complete infinitesimal generator plays a central role in the study of structure of semigroups. (See [2, Chaps. XI and XII; 4; 5]. We note here that the complete infinitesimal generator is the same as the infinitesimal generator in the sense of [2].)

In [3], Miyadera and Tanaka studied the generation of a semigroup $\{T(t) : t \geqslant 0\}$ on X satisfying two conditions

(a) $T(t)C = CT(t)$ for $t > 0$,

(b) $R(C) \subset \Sigma := \{x \in X : \lim_{t \to 0^+} T(t)x = x\}$,

where C is an injective operator in $B(X)$ with dense range, via the theory of exponentially bounded C-semigroups. The set Σ is called the *continuity set of* $\{T(t): t \geqslant 0\}$. This result was well applied to obtain the generation theorems for semigroups of growth order α and of class $(C_{(k)})$ (see [3, Sects. 4 and 5]), because on condition that C is an injective operator in $B(X)$ with dense range, conditions (a) and (b) imply the following properties (c) and (d) which must be satisfied by semigroups of growth order α and of class $(C_{(k)})$ (see Definitions 3.1 and 3.2):

(c) $\bigcup_{t>0} T(t)[X]$ is dense in X,

(d) if $T(t)x = 0$ for every $t > 0$ then $x = 0$.

Our purpose in this paper is to establish the theory for the adjoint of a semigroup $\{T(t): t \geqslant 0\}$ on X satisfying two conditions (a) and (b) and then to apply the results to semigroups of growth order α and of class $(C_{(k)})$. To attain our object, we must determine the subspace X^+ of X^* so that the restrictions of $T(t)^*$ and C^* to X^+, denoted by $T(t)^+$ and C^+, respectively, satisfy four properties corresponding to conditions (a) through (d) stated above. From the fact that the semigroup $\{T(t)^+ : t \geqslant 0\}$ on X^+ satisfies the property corresponding to condition (c) if and only if the continuity set of $T(t)^+$ contains a dense subset of X^+ and the assertion by Lemma 2.1 below that for $x^* \in C^*\overline{D(A^*)}$, $T(t)^* x^*$ is continuous in $t \geqslant 0$, it is reasonable to take the closure $\overline{C^*D(A^*)}$ of $C^*\overline{D(A^*)}$ as X^+.

Our result in Section 2 is stated as follows (see Theorem 2.3).

Let A be the complete infinitesimal generator of a semigroup $\{T(t) : t \geqslant 0\}$ satisfying two conditions (a) and (b). Set $X^+ = \overline{C^*D(A^*)}$ and define $T(t)^+$ and C^+ by the restrictions of $T(t)^*$ and C^* to X^+, respectively. Then $\{T(t)^+ : t \geqslant 0\}$ is a semigroup on X^+ satisfying four properties

(a^+) $T(t)^+ C^+ = C^+ T(t)^+$ for $t > 0$,

(b^+) $R(C^+) \subset \{x^* \in X^+ : \lim_{t \to 0^+} T(t)^+ x^* = x^*\}$,

(c^+) $\bigcup_{t>0} T(t)^+ [X^+]$ is dense in X^+,

(d^+) if $T(t)^+ x^* = 0$ for every $t > 0$ then $x^* = 0$, and if $R(C^+)$ is dense in X^+ then its complete infinitesimal generator is equal to $\overline{A^+|_{C^+D(A^+)}}$, where A^+ is the part of A^* in X^+. In particular, if one takes the identity as the operator C then the above result is the well known theorem due to Phillips [7] (see Corollary 2.5).

Finally, Section 3 is devoted to applications of our results in Section 2 to semigroups of growth order α and of class $(C_{(k)})$.

2. Adjoints of Semigroups

Let C be an injective operator in $B(X)$ with dense range. In this section we study the adjoint of a semigroup $\{T(t) : t \geqslant 0\}$ satisfying two conditions

(a) $T(t)C = CT(t)$ for $t > 0$,

(b) $R(C) \subset \Sigma := \{x \in X : \lim_{t \to 0^+} T(t)x = x\}$.

By [3, Theorem 2.1] we see that the complete infinitesimal generator A of $\{T(t) : t \geqslant 0\}$ exists and is densely defined in X, and $\{CT(t) : t \geqslant 0\}$ is an exponentially bounded C-semigroup on X whose complete infinitesimal generator is equal to A. We refer to [1, 3, 8] for further information on exponentially bounded C-semigroups and semigroups satisfying two conditions (a) and (b).

We start with

Lemma 2.1. (i) *The set* $\overline{C^*D(A^*)}$ *is invariant under* $T(t)^*$ *and* C^*.

(ii) *For* $x^* \in \overline{C^*D(A^*)}$, $T(t)^* x^*$ *is continuous in* $t \geqslant 0$.

Proof. Let A_0 be the infinitesimal generator of $\{T(t) : t \geqslant 0\}$. Then it is easy to see that $T(t)x \in D(A_0)$ and $A_0 T(t)x = T(t) A_0 x$ for $x \in D(A_0)$ and $t \geqslant 0$. By the definition of the complete infinitesimal generator A of $\{T(t) : t \geqslant 0\}$ we have $T(t)x \in D(A)$ and $AT(t)x = T(t) Ax$ for $x \in D(A)$ and $t \geqslant 0$. Let $x \in D(A)$, $x^* \in D(A^*)$, and $t \geqslant 0$. Then, by the fact shown above we have $\langle Ax, T(t)^* x^* \rangle = \langle T(t) Ax, x^* \rangle = \langle AT(t)x, x^* \rangle = \langle x, T(t)^* A^* x^* \rangle$, which shows that $T(t)^* x^* \in D(A^*)$ and $A^* T(t)^* x^* = T(t)^* A^* x^*$. Combining this fact with the relation $T(t)^* C^* = C^* T(t)^*$ we have $T(t)^* [C^* D(A^*)] = C^* T(t)^* (D(A^*)) \subset C^* D(A^*)$ for $t \geqslant 0$. Since A is the complete infinitesimal generator of an exponentially bounded C-semigroup $\{CT(t) : t \geqslant 0\}$ on X we see that $Cx \in D(A)$ and $ACx = CAx$ for $x \in D(A)$, from which it easily follows that $C^* D(A^*) \subset D(A^*)$. We deduce from these facts that the desired claim (i) holds.

To prove (ii), let $x^* \in D(A^*)$. Then we have for $x \in X$ and $t, s \geqslant 0$,

$$\begin{aligned}\langle x, T(t)^* C^* x^* - T(s)^* C^* x^* \rangle &= \langle T(t) Cx - T(s) Cx, x^* \rangle \\ &= \left\langle A \int_s^t T(r) Cx \, dr, x^* \right\rangle \\ &= \left\langle \int_s^t T(r) Cx \, dr, A^* x^* \right\rangle. \qquad (2.1)\end{aligned}$$

By this equality we find

$$\|T(t)^* C^*x^* - T(s)^* C^*x^*\| \leqslant \|A^*x^*\| \, |t-s| \sup\{\|T(r)C\| : r \in [0, T]\}$$

for $t, s \in [0, T]$ and $x^* \in D(A^*)$. Noting that $\|T(t)^* C^*\|$ is bounded on $[0, T]$ we obtain the assertion (ii). Q.E.D.

LEMMA 2.2. (i) *For* $x^* \in \overline{D(A^*)}$, $T(t)^* x^*$ *is continuous in* $t > 0$.

(ii) *For* $x^* \in \overline{D(A^*)}$ *and* $t, s > 0$,

$$\int_s^t T(r)^* x^* \, dr \in D(A^*)$$

and

$$A^* \left(\int_s^t T(r)^* x^* \, dr \right) = T(t)^* x^* - T(s)^* x^*.$$

Proof. To prove (i), let $x^* \in D(A^*)$ and $\varepsilon > 0$. Then we find by (2.1)

$$\begin{aligned} &|\langle Cx, T(t)^* x^* - T(s)^* x^* \rangle| \\ &\quad \leqslant \|A^*x^*\| \, |t-s| \sup\{\|T(r)\| : r \in [\varepsilon, 1/\varepsilon]\} \, \|Cx\| \end{aligned}$$

for $t, s \in [\varepsilon, 1/\varepsilon]$ and $x \in X$. Since $R(C)$ is dense in X we see that $T(t)^* x^* : [\varepsilon, 1/\varepsilon] \to X^*$ is Lipschitz continuous. Noting that $\|T(t)^*\|$ is bounded on every compact subinterval of $(0, \infty)$ we obtain the assertion (i).

To prove (ii), let $x \in D(A)$, $x^* \in \overline{D(A^*)}$, and $t, s > 0$. Since $CD(A)$ is a core of A (see [8, Theorem 2.1]) there exists a sequence $\{x_n\}$ in $D(A)$ such that $Cx_n \to x$ and $ACx_n \to Ax$ as $n \to \infty$. Then we have

$$\begin{aligned} \left\langle ACx_n, \int_s^t T(r)^* x^* \, dr \right\rangle &= \int_s^t \langle T(r) ACx_n, x^* \rangle \, dr \\ &= \langle T(t) Cx_n - T(s) Cx_n, x^* \rangle \\ &= \langle Cx_n, T(t)^* x^* - T(s)^* x^* \rangle. \end{aligned}$$

Passing to the limit as $n \to \infty$ we have

$$\left\langle Ax, \int_s^t T(r)^* x^* \, dr \right\rangle = \langle x, T(t)^* x^* - T(s)^* x^* \rangle,$$

which shows the desired claim (ii). Q.E.D.

The main result of this section is given by

THEOREM 2.3. *Let C be an injective bounded linear operator with dense range. Let A be the complete infinitesimal generator of a semigroup $\{T(t) : t \geqslant 0\}$ on X satisfying two conditions*

(a) $T(t)C = CT(t)$ *for* $t > 0$,

(b) $R(C) \subset \Sigma := \{x \in X : \lim_{t \to 0^+} T(t)x = x\}$.

Set $X^+ = \overline{C^*D(A^*)}$ *and define* $T(t)^+$ *and* C^+ *by the restrictions of* $T(t)^*$ *and* C^* *to* X^+, *respectively. Then we have*:

(i) C^+ *is an injective operator in* $B(X^+)$ *and* $\{T(t)^+ : t \geqslant 0\}$ *is a semigroup on* X^+ *satisfying four properties*

(a^+) $T(t)^+ C^+ = C^+ T(t)^+$ *for* $t > 0$,

(b^+) $R(C^+) \subset \{x^* \in X^+ : \lim_{t \to 0^+} T(t)^+ x^* = x^*\}$,

(c^+) $\bigcup_{t > 0} T(t)^+ [X^+]$ *is dense in* X^+,

(d^+) *if* $T(t)^+ x^* = 0$ *for every* $t > 0$ *then* $x^* = 0$;

(ii) *The infinitesimal generator* A_0^+ *of* $\{T^+(t) : t \geqslant 0\}$ *is a densely defined closable linear operator in* X^+ *and*

$$\overline{A_0^+|_{C^+D(\overline{A_0^+})}} \subset \overline{A^+|_{C^+D(A^+)}} \subset \overline{A_0^+} \subset A^+,$$

where A^+ *is the part of* A^* *in* X^+;

(iii) *If* $R(C^+)$ *is dense in* X^+ *then* $\overline{A^+|_{C^+D(A^+)}}$ *is the complete infinitesimal generator of* $\{T(t)^+ : t \geqslant 0\}$ *on* X^+.

Proof. By (i) of Lemma 2.1 we see that $T(t)^+$ and C^+ are operators in $B(X^+)$. It is obvious that C^+ is injective. By (i) of Lemma 2.2 and the fact that $\{T(t)^* : t \geqslant 0\}$ has the semigroup property (1.1) we see that $\{T(t)^+ : t \geqslant 0\}$ is a semigroup on X^+. The property (a^+) follows from the condition (a). The property (b^+) is obtained by combining (ii) of Lemma 2.1 and the inclusion relation $C^*D(A^*) \subset D(A^*)$. By (ii) of Lemma 2.1 we see that Σ^+ is dense in X^+, where Σ^+ is the continuity set of $\{T(t)^+ : t \geqslant 0\}$. Combining this fact with the fact that $\bigcup_{t > 0} T(t)^+ [X^+]$ is dense in Σ^+ we obtain the property (c^+). To show (d^+), let $T(t)^+ x^* = 0$ for $t > 0$. Then we have $0 = C^+ T(t)^+ x^* = T(t)^+ C^+ x^*$ for $t > 0$. Letting $t \to 0+$ we obtain $C^+ x^* = 0$ because of the property (b^+). The injectivity of C^+ implies $x^* = 0$.

We prove (ii). It is easy to see that for $x^* \in \Sigma^+$, $\int_0^t T(s)^+ x^* \, ds \in D(A_0^+)$ for $t \geqslant 0$ and

$$A_0^+ \left(\int_0^t T(s)^+ x^* \, ds \right) = T(t)^+ x^* - x^* \tag{2.2}$$

for $t \geq 0$. Since $\lim_{t \to 0+} t^{-1} \int_0^t T(s)^+ x^* \, ds = x^*$ for $x^* \in \Sigma^+$, we see that $D(A_0^+)$ is dense in the dense subset Σ^+ of X^+, and so $D(A_0^+)$ is dense in X^+. To prove that A_0^+ is closable, let $x^* \in D(A_0^+)$. Then, since $x^* \in X^+ \subset \overline{D(A^*)}$ and $T(t)^+ x^* : [0, \infty) \to X^+$ is continuous, we find by (ii) of Lemma 2.2

$$T(t)^+ x^* - x^* = A^* \left(\int_0^t T(r)^+ x^* \, dr \right)$$

for $t \geq 0$. Dividing this equality by $t > 0$ and then passing to the limit as $t \to 0+$ we obtain the relation $A_0^+ \subset A^+$, and so A_0^+ is closable and $\overline{A_0^+} \subset A^+$. By the property (a^+) and the definition of $\overline{A_0^+}$ we have $C^+ D(\overline{A_0^+}) \subset D(\overline{A_0^+})$. We deduce from this fact and the inclusion relation $\overline{A_0^+} \subset A^+$ that

$$\overline{A_0^+}|_{C^+ D(\overline{A_0^+})} \subset A^+|_{C^+ D(A^+)}.$$

To show $A^+|_{C^+ D(A^+)} \subset \overline{A_0^+}$, let $x^* \in C^+ D(A^+)$. Then we have $x^* \in \Sigma^+$ and $A^+ x^* = C^+ A^+ (C^{-1})^+ x^* \in \Sigma^+$. Combining (2.2) and (ii) of Lemma 2.2 we find

$$\overline{A_0^+} \left(\frac{1}{t} \int_0^t T(r)^+ x^* \, dr \right) = \frac{1}{t} \int_0^t T(r)^+ A^+ x^* dr$$

and as $t \to 0+$ the right-hand side tends to $A^+ x^*$ and $(1/t) \int_0^t T(r)^+ x^* \, dr \to x^*$. This proves $A^+|_{C^+ D(A^+)} \subset \overline{A_0^+}$, and so (ii) is proved.

We demonstrate (iii). Using the assumption that $R(C^+)$ is dense in X^+, we see by [3, Theorem 2.1, 8, Theorem 2.1] that the complete infinitesimal generator $\overline{A_0^+}$ of $\{T(t)^+ : t \geq 0\}$ coincides with that of the exponentially bounded C^+-semigroup $\{C^+ T(t)^+ : t \geq 0\}$ and then $C^+ D(\overline{A_0^+})$ is a core of $\overline{A_0^+}$. The assertion (iii) follows from this property and (ii). Q.E.D.

COROLLARY 2.4. *Let A be the complete infinitesimal generator of a semigroup $\{T(t) : t \geq 0\}$ on X satisfying two conditions* (a) *and* (b) *in Theorem* 2.3. *If X is reflexive then $\{T(t)^* : t \geq 0\}$ is a semigroup on X^* satisfying four properties*

- (a^*) $T(t)^* C^* = C^* T(t)^*$ *for* $t > 0$,
- (b^*) $R(C^*) \subset \{x^* \in X^* : \lim_{t \to 0+} T(t)^* x^* = x^*\}$,
- (c^*) $\bigcup_{t > 0} T(t)^* [X^*]$ *is dense in* X^*,
- (d^*) *if* $T(t)^* x^* = 0$ *for* $t > 0$ *then* $x^* = 0$,

and $\overline{A^*|_{C^* D(A^*)}}$ *is the complete infinitesimal generator of* $\{T(t)^* : t \geq 0\}$.

Proof. The reflexivity of X implies that $D(A^*)$ and $R(C^*)$ are dense in X^*, and so the set $C^*D(A^*)$ is dense in X^*, that is, $X^+ = X^*$. Corollary 2.4 immediately follows from Theorem 2.3. Q.E.D.

Remark. Let C be an injective operator in $B(X)$ with dense range. Suppose that X is reflexive. Then we may prove that if $\{S(t) : t \geqslant 0\}$ is an exponentially bounded C-semigroup on X and if A and Z are the complete infinitesimal generator and the generator of $\{S(t) : t \geqslant 0\}$, respectively, then $\{S(t)^* : t \geqslant 0\}$ is an exponentially bounded C^*-semigroup on X^* and A^* and Z^* are the generator and the complete infinitesimal generator of $\{S(t)^* : t \geqslant 0\}$, respectively. This fact shows that the complete infinitesimal generator of $\{T(t)^* : t \geqslant 0\}$ in Corollary 2.4 coincides with the adjoint of the generator of the exponentially bounded C-semigroup $\{CT(t) : t \geqslant 0\}$ on X.

Taking the identity operator I on X as the operator C in Theorem 2.3 and Corollary 2.4 we have

COROLLARY 2.5 [6, 7]. *Let A be the infinitesimal generator of a semigroup $\{T(t) : t \geqslant 0\}$ of class (C_0) on X. Then we have:*

(i) $\{T(t)^*|_{\overline{D(A^*)}} : t \geqslant 0\}$ *is a semigroup of class (C_0) on $\overline{D(A^*)}$ whose infinitesimal generator is the part of A^* in $\overline{D(A^*)}$;*

(ii) *If X is reflexive then $\{T(t)^* : t \geqslant 0\}$ is a semigroup of class (C_0) on X^* whose infinitesimal generator is A^*.*

3. APPLICATIONS

Applying the results in the previous section we deal with the adjoints of semigroups of growth order α and the adjoints of semigroups of class $(C_{(k)})$ in a unified way.

If $\{T(t) : t \geqslant 0\}$ is a semigroup on X then $\omega_0 := \lim_{t \to \infty} t^{-1} \log \|T(t)\|$ exists and $-\infty \leqslant \omega_0 < \infty$. (See [2, p. 306].) This ω_0 is called the *type* of the semigroup.

We start with the definition of semigroups of growth order α.

DEFINITION 3.1. Let $\alpha \geqslant 0$. A semigroup $\{T(t) : t \geqslant 0\}$ on X is said to be *of growth order* α if it satisfies the following three conditions:

(i) $X_0 := \bigcup_{t>0} T(t)[X]$ is dense in X;

(ii) If $T(t)x = 0$ for $t > 0$ then $x = 0$;

(iii) $t^\alpha \|T(t)\|$ is bounded as $t \to 0+$.

Every semigroup of growth order α has the complete infinitesimal generator. We refer to [5] for further information on semigroups of growth order α.

DEFINITION 3.2. Let k be a nonnegative integer. A semigroup $\{T(t) : t \geqslant 0\}$ on X is said to be *of class* $(C_{(k)})$ if it satisfies the following three conditions:

(i) $X_0 := \bigcup_{t>0} T(t)[X]$ is dense in X;

(ii) There exists an $\omega > \omega_0$ such that for each $\lambda > \omega$ there is an injective operator $R(\lambda) \in B(X)$ satisfying $R(\lambda)x = \int_0^\infty e^{-\lambda t}T(t)x\,dt$ for $x \in X_0$;

(iii) $D(A^k) \subset \Sigma$, where Σ is the continuity set of $\{T(t) : t \geqslant 0\}$ and A is the complete infinitesimal generator of $\{T(t) : t \geqslant 0\}$.

Note that (i) and (ii) of Definition 3.2 imply the existence of the complete infinitesimal generator A of $\{T(t) : t \geqslant 0\}$ and then $R(\lambda) = R(\lambda : A)$ (the resolvent of A at λ) for $\lambda > \omega$. We refer to [4] for further information on semigroups of class $(C_{(k)})$.

We first consider the adjoint of semigroup $\{T(t) : t \geqslant 0\}$ of growth order $\alpha > 0$ on X. Let $c > \omega_0$ and define $C \in B(X)$ by

$$Cx = \frac{1}{[\alpha]!} \int_0^\infty t^{[\alpha]} e^{-ct} T(t)x\,dt$$

for $x \in X$, where $[\alpha]$ denotes the integral part of α. Then it is known that C is an injective operator with dense range, and $\{T(t) : t \geqslant 0\}$ and C satisfy two conditions (a) and (b) in Theorem 2.3. (See [3, Example 2].)

THEOREM 3.1. *Let A be the complete infinitesimal generator of a semigroup $\{T(t) : t \geqslant 0\}$ of growth order α. Set $X^+ = \overline{D((A^*)^{[\alpha]+2})}$ and define $T(t)^+$ by the restriction of $T(t)^*$ to X^+. Then $\{T(t)^+ : t \geqslant 0\}$ is a semigroup of growth order α and its complete infinitesimal generator is equal to $\overline{A^+|_{D((A^*)^{[\alpha]+2})}}$, where A^+ is the part of A^* in X^+.*

Proof. Set $X^+ = \overline{C^*D(A^*)}$ and define $T(t)^+$ by the restriction of $T(t)^*$ to X^+. Then by Theorem 2.3 we see that $\{T(t)^+ : t \geqslant 0\}$ is a semigroup of growth order α on X^+.

We prove that X^+ coincides with the set $\overline{D((A^*)^{[\alpha]+2})}$. By [5, Lemma 4.1] (see also the proof of [5, Lemma 3.4]) we have $C(c-A)^{[\alpha]+1}x = (c-A)^{[\alpha]+1}Cx = x$ for $x \in D(A^{[\alpha]+1})$, from which it follows that

$$\langle x, C^*(c-A^*)^{[\alpha]+1}x^* \rangle = \langle (c-A)^{[\alpha]+1}Cx, x^* \rangle = \langle x, x^* \rangle$$

for $x^* \in D((A^*)^{[\alpha]+1})$ and $x \in D(A^{[\alpha]+1})$. Since $D(A^{[\alpha]+1})$ is dense in X we have $C^*(c-A^*)^{[\alpha]+1} x^* = x^*$ for $x^* \in D((A^*)^{[\alpha]+1})$, which implies

$$D((A^*)^{[\alpha]+2}) \subset C^* D(A^*). \tag{3.1}$$

By (ii) of Lemmas 2.1 and 2.2 we find for $x^* \in D(A^*)$ and $t \geqslant 0$,

$$A^*\left(\int_0^t T(r)^* C^* x^* \, dr\right) = T(t)^* C^* x^* - C^* x^*,$$

from which it follows that for $k \geqslant 0$, $x^* \in D(A^*)$, and $t \geqslant 0$,

$$\begin{aligned} &\frac{1}{k!}\int_0^t (t-s)^k T(s)^* C^* x^* \, ds \\ &\quad = \frac{1}{k!}\int_0^t (t-s)^k \left[C^* x^* + A^* \int_0^s T(r)^* C^* x^* \, dr \right] ds \\ &\quad = \frac{t^{k+1}}{(k+1)!} C^* x^* + A^* \left(\frac{1}{(k+1)!} \int_0^t (t-s)^{k+1} T(s)^* C^* x^* \, ds \right). \end{aligned}$$

By this equality we have inductively for $k = 1, 2, 3, \ldots$,

$$\frac{1}{k!}\int_0^t (t-s)^k T(s)^* C^* x^* \, ds \in D((A^*)^{k+1}) \qquad \text{if} \quad x^* \in D((A^*)^k),$$

which implies the inclusion relation that $C^* D((A^*)^k) \subset \overline{D((A^*)^{k+1})}$ for $k = 1, 2, 3, \ldots$. Using this fact we obtain by induction

$$(C^*)^k D(A^*) \subset \overline{D((A^*)^{k+1})} \tag{3.2}$$

for $k = 0, 1, 2, \ldots$. Since the restriction C^+ of C^* to X^+ is written as

$$C^+ x^* = \frac{1}{[\alpha]!}\int_0^\infty t^{[\alpha]} e^{-ct} T(t)^+ x^* \, dt \tag{3.3}$$

for $x^* \in X^+$, [3, Example 2] asserts that $R(C^+)$ is dense in X^+, and so we have $\overline{(C^*)^k D(A^*)} = \overline{C^* D(A^*)}$ for $k = 1, 2, 3, \ldots$. Combining this relation, (3.2), and (3.1) we obtain the desired claim that $X^+ = \overline{D((A^*)^{[\alpha]+2})}$.

Finally we prove that $\overline{A^+|_{D((A^+)^{[\alpha]+2})}}$ is the complete infinitesimal generator of $\{T(t)^+ : t \geqslant 0\}$. Since $R(C^+)$ is dense in X^+, we see by (iii) of Theorem 2.3 that

$$\overline{A_0^+} = \overline{A^+|_{C^+ D(A^+)}}, \tag{3.4}$$

where A_0^+ is the infinitesimal generator of $\{T(t)^+ : t \geqslant 0\}$. Since $D((\overline{A_0^+})^{[\alpha]+2})$ is a core of $\overline{A_0^+}$ and $\overline{A_0^+}|_{D((\overline{A_0^+})^{[\alpha]+2})} \subset A^+|_{D((A^+)^{[\alpha]+2})}$ by (ii) of Theorem 2.3, we have $\overline{A_0^+} \subset \overline{A^+|_{D((A^+)^{[\alpha]+2})}}$. To prove the converse inclusion it suffices to show

$$C^+ D(A^+) \supset D((A^+)^{[\alpha]+2}) \tag{3.5}$$

because of (3.4). By (ii) of Lemma 2.2 we find $(d/dt)\, T(t)^+ x^* = T(t)^+ A^+ x^*$ for $t > 0$ and $x^* \in D(A^+)$. Similarly as in the proof of [5, Lemma 3.2] we find

$$\begin{aligned} 0 &= T(t)^+ x^* - \frac{1}{[\alpha]!} \int_0^\infty s^{[\alpha]} e^{-cs} T(t+s)^+ (c + A^+)^{[\alpha]+1} x^* \, ds \\ &= T(t)^+ \left(x^* - \frac{1}{[\alpha]!} \int_0^\infty s^{[\alpha]} e^{-cs} T(s)^+ (c - A^+)^{[\alpha]+1} x^* \, ds \right) \end{aligned}$$

for $t > 0$ and $x^* \in D((A^+)^{[\alpha]+1})$. Here we have used the fact that $t^\alpha \|T(t)^+\|$ is bounded as $t \to 0+$ to obtain the last equality. Since $\{T(t)^+ : t \geqslant 0\}$ satisfies the property (d$^+$) we have

$$x^* = \frac{1}{[\alpha]!} \int_0^\infty s^{[\alpha]} e^{-cs} T(s)^+ (c - A^+)^{[\alpha]+1} x^* \, ds$$

for $x^* \in D((A^+)^{[\alpha]+1})$. Combining this equality and (3.3) we obtain the inclusion relation (3.5). Q.E.D.

COROLLARY 3.2. *Let A be the complete infinitesimal generator of a semigroup $\{T(t) : t \geqslant 0\}$ of growth order $\alpha > 0$ on X. If X is reflexive then $\{T(t)^* : t \geqslant 0\}$ is a semigroup of growth order α on X^* whose complete infinitesimal generator is equal to $\overline{A^*|_{D((A^*)^{[\alpha]+2})}}$.*

We next consider the adjoint of a semigroup $\{T(t) : t \geqslant 0\}$ of class $(C_{(k)})$, where k is a nonnegative integer. If A is the complete infinitesimal generator of $\{T(t) : t \geqslant 0\}$ then by the definition there exists an ω $(> \omega_0)$ such that $(\omega, \infty) \subset \rho(A)$ and

$$R(\lambda : A)x = \int_0^\infty e^{-\lambda t} T(t) x \, dt \tag{3.6}$$

for $x \in X_0$ and $\lambda > \omega$. Let $c > \omega$ and set $C = R(c : A)^k$. Then C is an injective operator in $B(X)$ with dense range, and $\{T(t) : t \geqslant 0\}$ and C satisfy two conditions (a) and (b) in Theorem 2.3. (See [3, Example 1].)

THEOREM 3.3. *Let A be the complete infinitesimal generator of a semigroup $\{T(t) : t \geqslant 0\}$ of class $(C_{(k)})$. Set $X^+ = \overline{D((A^*)^{k+1})}$ and define $T(t)^+$*

by the restriction of $T(t)^*$ *to* X^+. *Then* $\{T(t)^+ : t \geqslant 0\}$ *is a semigroup of class* $(C_{(k)})$ *and its complete infinitesimal generator is equal to the part of* A^* *in* X^+.

Proof. Since $(\omega, \infty) \subset \rho(A^*)$ and $C^* = R(c:A^*)^k$ we have $\overline{C^*D(A^*)} = \overline{D((A^*)^{k+1})}$. Let A^+ be the part of A^* in X^+. Then we see that $D(A^+)$ is dense in X^+ by (ii) of Theorem 2.3 and that $c \in \rho(A^+)$ and $C^+ = R(c:A^+)^k$. It thus follows that $R(C^+)$ is dense in X^+ and $C^+D(A^+)$ is a core of A^+. Part (iii) of Theorem 2.3 asserts that A^+ is the complete infinitesimal generator of the semigroup $\{T(t)^+ : t \geqslant 0\}$ on X^+.

We prove that the semigroup $\{T(t)^+ : t \geqslant 0\}$ is of class $(C_{(k)})$ on X^+. The conditions (i) and (iii) of Definition 3.2 follow from the properties (c^+) and (b^+), respectively. To prove condition (ii) of Definition 3.2, let $\lambda > \omega$, $x \in X_0$, and $x^* \in \bigcup_{t>0} T(t)^+ [X^+]$. Then we find by (3.6)

$$\langle x, R(\lambda:A^+)x^* \rangle = \left\langle x, \int_0^\infty e^{-\lambda t} T(s)^+ x^* \, ds \right\rangle.$$

Since X_0 is dense in X, we see that $\{T(t)^+ : t \geqslant 0\}$ satisfies condition (ii) of Definition 3.2. Q.E.D.

COROLLARY 3.4. *Let* A *be the complete infinitesimal generator of a semigroup* $\{T(t) : t \geqslant 0\}$ *of class* $(C_{(k)})$ *on* X. *If* X *is reflexive then* $\{T(t)^* : t \geqslant 0\}$ *is a semigroup of class* $(C_{(k)})$ *on* X^* *whose complete infinitesimal generator is* A^*.

Remark. Let $\{T(t) : t \geqslant 0\}$ be one of semigroups of classes such as (A), $(1, A)$, and $(1, C_1)$ and A be the complete infinitesimal generator of $\{T(t) : t \geqslant 0\}$. Set $X^+ = \overline{D(A^*)}$ and define $T(t)^+$ by the restriction of $T(t)^*$ to X^+. Then, using Theorem 3.3 we may prove [2, Theorem 14.4.1 and Corollary] which asserts that two semigroups $\{T(t) : t \geqslant 0\}$ and $\{T(t)^+ : t \geqslant 0\}$ belong to the same class and the complete infinitesimal generator of $\{T(t)^+ : t \geqslant 0\}$ is the part of A^* in X^+.

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